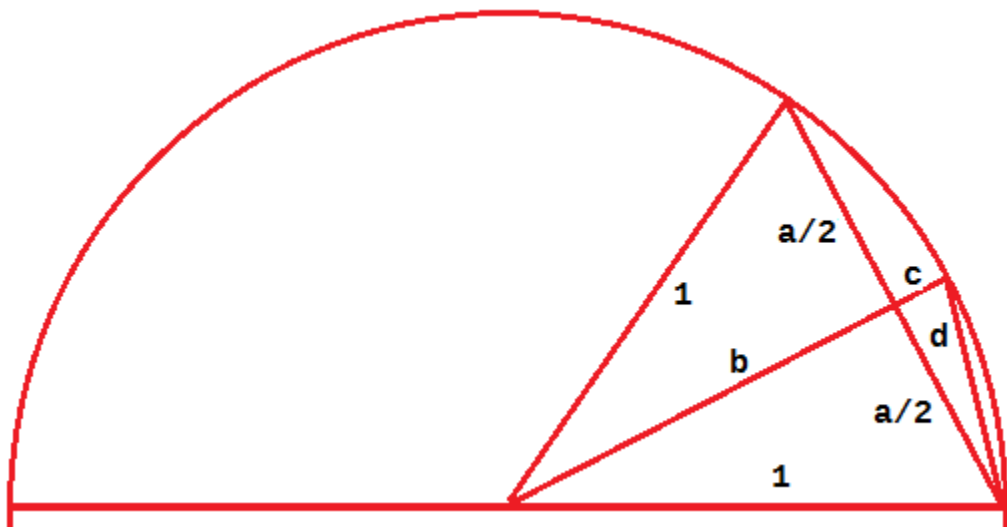


Transcendental Equations

Imprecise Precision in Mathematics

In mathematics there are certain formulas that are sometimes known as "transcendental" equations. A "transcendental" equation occurs when two fundamentally different systems are brought together.

The most well known of the "transcendental" equations is the solution for "pi" representing the ratio between the circumference of a circle and the diameter of a circle. The actual process is vary simple. A circle is first divided into six equilateral triangles with by definition each of the six sides having a length equal to the radius. Then a line is drawn from the center to a point on the circumference of the circle. This line bisects one of the equilateral triangles. Now a new chord is drawn between the original point on the circumference and the new point of bisection. Using the Pythagorean theorem, the length of this new chord is determined. It will be shorter that the original chord, but there will be twice as many. This procedure is repeated as many times as needed. Here is an illustration of this division.



Here we have the first level of division worked out for the calculation of pi with respect to the preceding illustration.

$$\mathbf{a = 1}$$

$$\mathbf{b = \sqrt{1 - \left(\frac{a}{2}\right)^2} = \sqrt{1 - \left(\frac{1}{2}\right)^2}}$$

$$\mathbf{c = 1 - b = 1 - \sqrt{1 - \left(\frac{a}{2}\right)^2} = 1 - \sqrt{1 - \left(\frac{1}{2}\right)^2}}$$

$$\mathbf{d = \sqrt{\left(\frac{a}{2}\right)^2 + c^2} = \sqrt{\left(\frac{a}{2}\right)^2 + \left[1 - \sqrt{1 - \left(\frac{a}{2}\right)^2}\right]^2}}$$

$$\mathbf{d = \sqrt{\left(\frac{1}{2}\right)^2 + \left[1 - \sqrt{1 - \left(\frac{1}{2}\right)^2}\right]^2} = \sqrt{\frac{1}{4} + 1 - \sqrt{3} + 1 - \frac{1}{4}} = \sqrt{2 - \sqrt{3}}}$$

$$\mathbf{p = 2 \cdot 3 \cdot \sqrt{2 - \sqrt{3}} = 3.106}$$

Let us take a quick look at the second stage.

$$\mathbf{a = d = \sqrt{2 - \sqrt{3}}}$$

$$\mathbf{b = \sqrt{1 - \left(\frac{\sqrt{2 - \sqrt{3}}}{2}\right)^2} = \sqrt{1 - \frac{2 - \sqrt{3}}{4}} = \sqrt{1 - \frac{1}{2} + \frac{\sqrt{3}}{4}} = \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}}$$

$$\mathbf{c = 1 - \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}}$$

$$\mathbf{d = \sqrt{\left(\frac{\sqrt{2 - \sqrt{3}}}{2}\right)^2 + \left(1 - \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}\right)^2} = \sqrt{2 - \sqrt{2 + \sqrt{3}}}}$$

$$\mathbf{p = 4 \cdot 3 \cdot \sqrt{2 - \sqrt{2 + \sqrt{3}}}} = 3.133$$

At the third stage there is another twist which carries through the entire pattern. The next sequence will show this final twist to the process for calculating for pi.

$$n-0: \pi = 2^0 \cdot 3 = 3$$

$$n-1: \pi = 2^1 \cdot 3 \cdot \sqrt{2 - \sqrt{3}} = 3.106$$

$$n-2: \pi = 2^2 \cdot 3 \cdot \sqrt{2 - \sqrt{2 + \sqrt{3}}} = 3.133$$

$$n-3: \pi = 2^3 \cdot 3 \cdot \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{3}}}} = 3.139$$

$$n-4: \pi = 2^4 \cdot 3 \cdot \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}} = 3.141$$

Here is a simple LIBERTY BASIC v4.03 computer program indicating the utilizing the procedure.

```

REM LIBERTY BASIC v4.03 Program
REM Imprecise Precision in Mathematics.bas
INPUT "Enter number of steps for pi: "; t
PRINT "n = 0      pi = 3.00000000"
  LET n = 1
  LET a = 1
WHILE n <= t
  LET b = (1 - (a/2)^2)^(1/2)
  LET c = 1 - b
  LET d = ((a/2)^2 + c^2)^(1/2)
  LET p = 2^n * 3 * d
  PRINT "n = "; using("##",n); "      pi = "; using("#.#####",p)
  LET n = n + 1
  LET a = d
WEND
END

```

```

Enter number of steps for pi: 14
n = 0      pi = 3.00000000
n = 1      pi = 3.10582854
n = 2      pi = 3.13262861
n = 3      pi = 3.13935020
n = 4      pi = 3.14103195
n = 5      pi = 3.14145247
n = 6      pi = 3.14155761
n = 7      pi = 3.14158389
n = 8      pi = 3.14159046
n = 9      pi = 3.14159211
n = 10     pi = 3.14159252
n = 11     pi = 3.14159262
n = 12     pi = 3.14159265
n = 13     pi = 3.14159265
n = 14     pi = 3.14159265

```

There are three problems with this program. Here are the problems in order.

[1] The first problem is that the calculation precision is dependent on the number of digits that the computer can use at any one time. If an abacus were used instead, the same issue would apply to the number of bead rows. Pen and paper might do better, but the procedure would become too tedious leading to transcription errors.

[2] The second problem is the need to round out the solution. This is an endless cycle that cuts inside of the circle. The natural solution is to always round out short. However, sometimes it is rounded high. Pi is used in trigonometry. Near the cardinal points is is vital to know if it is rounded low or rounded high.

[3] The third problem is that the resulting irregular square roots are solved by taking the natural log of the number, dividing it by two, and then taking the natural anti-log of the result in order to find the length of the new chord. Both the prerequisite natural log and the prerequisite anti-log are also "transcendental" equations, each solved by an infinite sequence.

Let us consider the Taylor infinite series for determining the natural log of a number. This is written in the handbook as:

$$\ln(x) = 2 \cdot \left[\frac{1}{1} \cdot \left(\frac{x-1}{x+1} \right)^1 + \frac{1}{3} \cdot \left(\frac{x-1}{x+1} \right)^3 + \frac{1}{5} \cdot \left(\frac{x-1}{x+1} \right)^5 \right] + \dots \quad x > 0$$

The natural logarithms have a base of 2.71828183. That is to say, $\ln(2.71828183) = 1$. Thus $\ln(2.71828183)^2 = 2$, and so on. When we square a number, we take the log of the number, and then we multiply

the log by two in order to obtain the log of the of the square. Alternately, when we take the square root of a number, we take the log of the number, and then we divide the log by two in order to obtain the log of the of the square root. The question here is just how precise is this procedure. This is important because when we calculate for pi, we have one transcendental equation feeding another transcendental equation in order to just process the logarithms.

Here is a little LIBERTY BASIC v4.03 program written to evaluate the relative precision of the applicable transcendental equation.

```

REM LIBERTY BASIC v4.03 Program
REM Taylor Natural Log.bas
INPUT "Enter number of terms: "; t
INPUT "Enter a value for x : "; x
LET n = 1
LET d = 0
WHILE n <= t
LET a = (x - 1)/(x + 1)
LET b = 2 * n - 1
LET c = d + (1/b) * (a^b)
LET e = 2 * c
PRINT "n = "; using("##", n); "    ln(x) = "; using("#.#####", e)
LET n = n + 1
LET d = c
WEND
END

```

In this program, the number of terms and the value of "x" is entered. The program returns the calculated value of $\ln(x)$ for each of the number of terms in the sequence. Let us examine some of the returns.

<pre> Enter number of terms: 20 Enter a value for x : 2.7182 n = 1 ln(x) = 0.92421064 n = 2 ln(x) = 0.98999636 n = 3 ln(x) = 0.99842514 n = 4 ln(x) = 0.99971078 n = 5 ln(x) = 0.99992431 n = 6 ln(x) = 0.99996162 n = 7 ln(x) = 0.99996836 n = 8 ln(x) = 0.99996961 n = 9 ln(x) = 0.99996984 n = 10 ln(x) = 0.99996989 n = 11 ln(x) = 0.99996989 n = 12 ln(x) = 0.99996990 n = 13 ln(x) = 0.99996990 n = 14 ln(x) = 0.99996990 n = 15 ln(x) = 0.99996990 n = 16 ln(x) = 0.99996990 n = 17 ln(x) = 0.99996990 n = 18 ln(x) = 0.99996990 n = 19 ln(x) = 0.99996990 n = 20 ln(x) = 0.99996990 </pre>	<pre> Enter number of terms: 20 Enter a value for x : 7.3891 n = 1 ln(x) = 1.52319081 n = 2 ln(x) = 1.81768836 n = 3 ln(x) = 1.92017837 n = 4 ln(x) = 1.96264054 n = 5 ln(x) = 1.98179660 n = 6 ln(x) = 1.99088746 n = 7 ln(x) = 1.99534918 n = 8 ln(x) = 1.99759205 n = 9 ln(x) = 1.99873993 n = 10 ln(x) = 1.99933564 n = 11 ln(x) = 1.99964826 n = 12 ln(x) = 1.99981383 n = 13 ln(x) = 1.99990217 n = 14 ln(x) = 1.99994962 n = 15 ln(x) = 1.99997525 n = 16 ln(x) = 1.99998915 n = 17 ln(x) = 1.99999673 n = 18 ln(x) = 2.00000087 n = 19 ln(x) = 2.00000314 n = 20 ln(x) = 2.00000439 </pre>
<pre> Enter number of terms: 20 Enter a value for x : 20.0855 n = 1 ln(x) = 1.81029618 n = 2 ln(x) = 2.30468387 n = 3 ln(x) = 2.54771292 n = 4 ln(x) = 2.68993578 n = 5 ln(x) = 2.78056417 n = 6 ln(x) = 2.84131515 n = 7 ln(x) = 2.88343065 n = 8 ln(x) = 2.91333493 n = 9 ln(x) = 2.93495290 n = 10 ln(x) = 2.95079999 n = 11 ln(x) = 2.96254688 n = 12 ln(x) = 2.97133415 n = 13 ln(x) = 2.97795755 n = 14 ln(x) = 2.98298209 n = 15 ln(x) = 2.98681476 n = 16 ln(x) = 2.98975226 n = 17 ln(x) = 2.99201307 n = 18 ln(x) = 2.99375949 n = 19 ln(x) = 2.99511298 n = 20 ln(x) = 2.99616502 </pre>	<pre> Enter number of terms: 20 Enter a value for x : 54.5982 n = 1 ln(x) = 1.92805522 n = 2 ln(x) = 2.52533411 n = 3 ln(x) = 2.85838251 n = 4 ln(x) = 3.07946700 n = 5 ln(x) = 3.23927288 n = 6 ln(x) = 3.36078554 n = 7 ln(x) = 3.45633974 n = 8 ln(x) = 3.53330254 n = 9 ln(x) = 3.59641311 n = 10 ln(x) = 3.64889100 n = 11 ln(x) = 3.69301650 n = 12 ln(x) = 3.73045859 n = 13 ln(x) = 3.76247162 n = 14 ln(x) = 3.79001911 n = 15 ln(x) = 3.81385475 n = 16 ln(x) = 3.83457724 n = 17 ln(x) = 3.85266849 n = 18 ln(x) = 3.86852084 n = 19 ln(x) = 3.88245686 n = 20 ln(x) = 3.89474411 </pre>

The upper left return represents the case for e^1 . The upper right represents the case for e^2 . The lower left represents the case for e^3 . The lower right represents the case for e^4 . Observe the four place imprecision, (because the number was entered in four places).

The logarithms are a tool of convenience. We may readily multiply and divide by simply adding and subtracting logarithms. The slide rule was specifically designed to add and subtract logarithms. Irrational powers of a number may be facilitated by the use of logarithms. If a logarithm is multiplied by another number, the product is the exponent or root of the number as the case may be. Thus, when the logarithm of a number is multiplied by 2, the product is the square of the number. Alternately, when the logarithm of a number is multiplied by 1/2, the product is the square root of the number.

This is all nice and dandy, but it is useless unless the resulting logarithm is converted back to the normal numbers by means of yet another transcendental equation. Thus for the simple apparent transcendental equation for pi, we have two other transcendental equations used multiple times. Let now consider the Taylor infinite series for e^x .

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad -\infty < x < \infty$$

Here is a little LIBERTY BASIC v4.03 program written to evaluate the relative precision of this particular transcendental equation.

```

REM LIBERTY BASIC v4.03 Program
REM Taylor Antilog.bas
INPUT "Enter number of terms : "; t
INPUT "Enter a value for ln(x): "; x
LET n = 1
LET a = 1
LET b = 1
WHILE n <= t
LET c = a + (x^n/b)
PRINT "n = "; using("##", n); "    ln(x) = "; using("#.#####", c)
LET n = n + 1
LET b = b * n
LET a = c
WEND
END

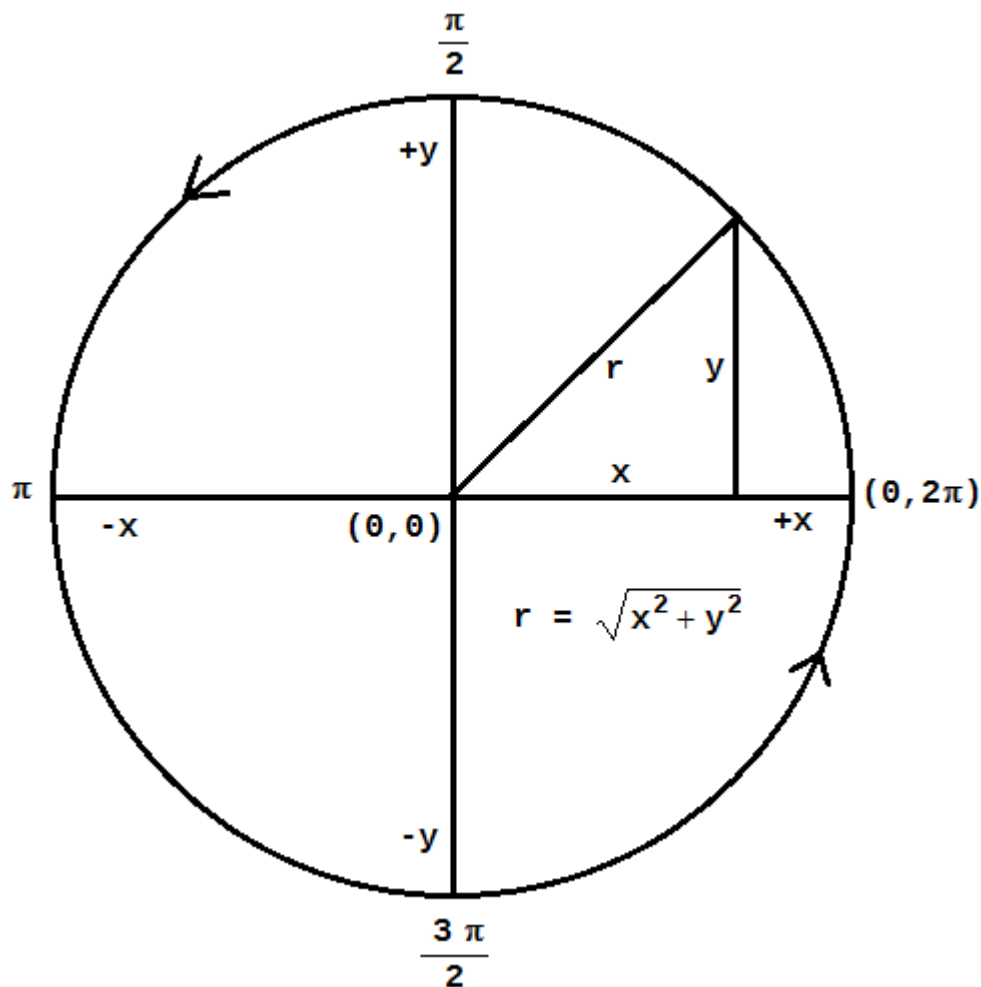
```

Here is a table arranged to show the antilogs of the preceding table in the same order.

Enter number of terms : 20 Enter a value for $\ln(x)$: 1 n = 1 $\ln(x) = 2.00000000$ n = 2 $\ln(x) = 2.50000000$ n = 3 $\ln(x) = 2.66666667$ n = 4 $\ln(x) = 2.70833333$ n = 5 $\ln(x) = 2.71666667$ n = 6 $\ln(x) = 2.71805556$ n = 7 $\ln(x) = 2.71825397$ n = 8 $\ln(x) = 2.71827877$ n = 9 $\ln(x) = 2.71828153$ n = 10 $\ln(x) = 2.71828180$ n = 11 $\ln(x) = 2.71828183$ n = 12 $\ln(x) = 2.71828183$ n = 13 $\ln(x) = 2.71828183$ n = 14 $\ln(x) = 2.71828183$ n = 15 $\ln(x) = 2.71828183$ n = 16 $\ln(x) = 2.71828183$ n = 17 $\ln(x) = 2.71828183$ n = 18 $\ln(x) = 2.71828183$ n = 19 $\ln(x) = 2.71828183$ n = 20 $\ln(x) = 2.71828183$	Enter number of terms : 20 Enter a value for $\ln(x)$: 2 n = 1 $\ln(x) = 3.00000000$ n = 2 $\ln(x) = 5.00000000$ n = 3 $\ln(x) = 6.33333333$ n = 4 $\ln(x) = 7.00000000$ n = 5 $\ln(x) = 7.26666667$ n = 6 $\ln(x) = 7.35555556$ n = 7 $\ln(x) = 7.38095238$ n = 8 $\ln(x) = 7.38730159$ n = 9 $\ln(x) = 7.38871252$ n = 10 $\ln(x) = 7.38899471$ n = 11 $\ln(x) = 7.38904602$ n = 12 $\ln(x) = 7.38905457$ n = 13 $\ln(x) = 7.38905588$ n = 14 $\ln(x) = 7.38905607$ n = 15 $\ln(x) = 7.38905610$ n = 16 $\ln(x) = 7.38905610$ n = 17 $\ln(x) = 7.38905610$ n = 18 $\ln(x) = 7.38905610$ n = 19 $\ln(x) = 7.38905610$ n = 20 $\ln(x) = 7.38905610$
Enter number of terms : 20 Enter a value for $\ln(x)$: 3 n = 1 $\ln(x) = 4.00000000$ n = 2 $\ln(x) = 8.50000000$ n = 3 $\ln(x) = 13.00000000$ n = 4 $\ln(x) = 16.37500000$ n = 5 $\ln(x) = 18.40000000$ n = 6 $\ln(x) = 19.41250000$ n = 7 $\ln(x) = 19.84642857$ n = 8 $\ln(x) = 20.00915179$ n = 9 $\ln(x) = 20.06339286$ n = 10 $\ln(x) = 20.07966518$ n = 11 $\ln(x) = 20.08410308$ n = 12 $\ln(x) = 20.08521256$ n = 13 $\ln(x) = 20.08546859$ n = 14 $\ln(x) = 20.08552346$ n = 15 $\ln(x) = 20.08553443$ n = 16 $\ln(x) = 20.08553649$ n = 17 $\ln(x) = 20.08553685$ n = 18 $\ln(x) = 20.08553691$ n = 19 $\ln(x) = 20.08553692$ n = 20 $\ln(x) = 20.08553692$	Enter number of terms : 20 Enter a value for $\ln(x)$: 4 n = 1 $\ln(x) = 5.00000000$ n = 2 $\ln(x) = 13.00000000$ n = 3 $\ln(x) = 23.66666667$ n = 4 $\ln(x) = 34.33333333$ n = 5 $\ln(x) = 42.86666667$ n = 6 $\ln(x) = 48.55555556$ n = 7 $\ln(x) = 51.80634921$ n = 8 $\ln(x) = 53.43174603$ n = 9 $\ln(x) = 54.15414462$ n = 10 $\ln(x) = 54.44310406$ n = 11 $\ln(x) = 54.54818021$ n = 12 $\ln(x) = 54.58320560$ n = 13 $\ln(x) = 54.59398264$ n = 14 $\ln(x) = 54.59706180$ n = 15 $\ln(x) = 54.59788291$ n = 16 $\ln(x) = 54.59808818$ n = 17 $\ln(x) = 54.59813648$ n = 18 $\ln(x) = 54.59814722$ n = 19 $\ln(x) = 54.59814948$ n = 20 $\ln(x) = 54.59814993$

As we can see from these returns, any precision value of pi involves an unimaginably tedious procedure.

In order to use the transcendental equations for the trigonometric functions it was first essential to define pi. In natural trigonometry the angles are expressed in terms of radians. If the radius of a circle is assigned a value of one, the circumference of the circle has a value of 2π . That is to say that the circumference of the circle is 2π times the radius of the circle. Thus, we get the term radians. This is a natural angle used and produced by the transcendental trigonometric equations.



$$\sin(\theta) = \frac{y}{\sqrt{x^2 + y^2}} \quad \cos(\theta) = \frac{x}{\sqrt{x^2 + y^2}} \quad \tan(\theta) = \frac{y}{x}$$

There are two transcendental equations of note for approximating the sine and the cosine respectively of an angle. In both cases the angle must be expressed in terms of radians. The transcendental equation for the tangent of an angle is more acrobatic. However, the tangent of an angle may readily be found by dividing the sine by the cosine. Both of the following transcendental equations are from the Taylor infinite series.

$$\sin(\theta) = \frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \frac{\theta^{11}}{11!} + \dots (-1)^{n+1} \cdot \frac{\theta^{2 \cdot n - 1}}{(2 \cdot n - 1)!}$$

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \frac{\theta^{10}}{10!} + \dots \quad 1 - (-1)^n \cdot \frac{\theta^{2 \cdot n}}{(2 \cdot n)!}$$

Here is a little program that demonstrates the sine function.

```

REM LIBERTY BASIC v4.03 program
REM taylorSine.bas
INPUT "Enter number of terms: "; t
INPUT "Enter an angle in deg: "; q
LET p = 3.14159265
LET r = q * (p/180)
LET n = 1 'begin count of terms after 1st term.
LET f = 1 '1st term factorial
LET b = 0
LET e = 1
WHILE n <= t
LET a = -1^(n + 1)
LET b = b + a * ((r^e)/f)
PRINT "n = "; using("##",n); " sin("; using("###",q); ") = ";
using("##.#####",b)
LET f = f * (e + 1) * (e + 2)
LET e = e + 2
LET n = n + 1
WEND
END

```

This program accepts the angle in degrees, converts it to radians, and then processes the value of the sine of the given angle.

Here is the cosine program. This program was not written using the given Taylor cosine infinite series, but rather a simple adjustment of the original given angle to shift it 90° out of phase using the Taylor infinite sine series.

```

REM LIBERTY BASIC v4.03 program
REM taylorCosine.bas
INPUT "Enter number of terms: "; t
INPUT "Enter an angle in deg: "; q
LET p = 3.14159265
LET r = (90 - q) * (p/180)
LET n = 1 'begin count of terms after 1st term.
LET f = 1 '1st term factorial
LET b = 0
LET e = 1
WHILE n <= t
LET a = -1^(n + 1)
LET b = b + a * ((r^e)/f)
PRINT "n = "; using("##",n); "      cos("; using("###",q); ") = ";
using("##.#####",b)
LET f = f * (e + 1) * (e + 2)
LET e = e + 2
LET n = n + 1
WEND
END

```

Let us examine some examples of these two small programs using the angles that will produce rational sines and cosines. In terms of degrees the following will apply:

$$\text{Sin}(0^\circ) = 0.00000000 \quad \text{Cos}(0^\circ) = 1.00000000$$

$$\text{Sin}(30^\circ) = 0.50000000$$

$$\text{Cos}(60^\circ) = 0.50000000$$

$$\text{Sin}(90^\circ) = 1.00000000$$

$$\text{Cos}(90^\circ) = 0.00000000$$

$$\text{Cos}(120^\circ) = -0.50000000$$

$$\text{Sin}(150^\circ) = 0.50000000$$

$$\text{Sin}(180^\circ) = 0.00000000$$

$$\text{Cos}(180^\circ) = -1.00000000$$

The conversion is $180^\circ = \pi$ rad.

Here are four discreet rational returns for the sine function.

<pre> Enter number of terms: 20 Enter an angle in deg: 30 n = 1 sin(30) = 0.52359878 n = 2 sin(30) = 0.49967418 n = 3 sin(30) = 0.50000213 n = 4 sin(30) = 0.49999999 n = 5 sin(30) = 0.50000000 n = 6 sin(30) = 0.50000000 n = 7 sin(30) = 0.50000000 n = 8 sin(30) = 0.50000000 n = 9 sin(30) = 0.50000000 n = 10 sin(30) = 0.50000000 n = 11 sin(30) = 0.50000000 n = 12 sin(30) = 0.50000000 n = 13 sin(30) = 0.50000000 n = 14 sin(30) = 0.50000000 n = 15 sin(30) = 0.50000000 n = 16 sin(30) = 0.50000000 n = 17 sin(30) = 0.50000000 n = 18 sin(30) = 0.50000000 n = 19 sin(30) = 0.50000000 n = 20 sin(30) = 0.50000000 </pre>	<pre> Enter number of terms: 20 Enter an angle in deg: 90 n = 1 sin(90) = 1.57079633 n = 2 sin(90) = 0.92483223 n = 3 sin(90) = 1.00452486 n = 4 sin(90) = 0.99984310 n = 5 sin(90) = 1.00000354 n = 6 sin(90) = 0.99999994 n = 7 sin(90) = 1.00000000 n = 8 sin(90) = 1.00000000 n = 9 sin(90) = 1.00000000 n = 10 sin(90) = 1.00000000 n = 11 sin(90) = 1.00000000 n = 12 sin(90) = 1.00000000 n = 13 sin(90) = 1.00000000 n = 14 sin(90) = 1.00000000 n = 15 sin(90) = 1.00000000 n = 16 sin(90) = 1.00000000 n = 17 sin(90) = 1.00000000 n = 18 sin(90) = 1.00000000 n = 19 sin(90) = 1.00000000 n = 20 sin(90) = 1.00000000 </pre>
<pre> Enter number of terms: 20 Enter an angle in deg: 150 n = 1 sin(150) = 2.61799388 n = 2 sin(150) = -0.37258064 n = 3 sin(150) = 0.65227309 n = 4 sin(150) = 0.48502936 n = 5 sin(150) = 0.50094978 n = 6 sin(150) = 0.49995781 n = 7 sin(150) = 0.50000139 n = 8 sin(150) = 0.49999997 n = 9 sin(150) = 0.50000000 n = 10 sin(150) = 0.50000000 n = 11 sin(150) = 0.50000000 n = 12 sin(150) = 0.50000000 n = 13 sin(150) = 0.50000000 n = 14 sin(150) = 0.50000000 n = 15 sin(150) = 0.50000000 n = 16 sin(150) = 0.50000000 n = 17 sin(150) = 0.50000000 n = 18 sin(150) = 0.50000000 n = 19 sin(150) = 0.50000000 n = 20 sin(150) = 0.50000000 </pre>	<pre> Enter number of terms: 20 Enter an angle in deg: 180 n = 1 sin(180) = 3.14159265 n = 2 sin(180) = -2.02612011 n = 3 sin(180) = 0.52404391 n = 4 sin(180) = -0.07522061 n = 5 sin(180) = 0.00692527 n = 6 sin(180) = -0.00044516 n = 7 sin(180) = 0.00002115 n = 8 sin(180) = -0.00000077 n = 9 sin(180) = 0.00000003 n = 10 sin(180) = 0.00000000 n = 11 sin(180) = 0.00000000 n = 12 sin(180) = 0.00000000 n = 13 sin(180) = 0.00000000 n = 14 sin(180) = 0.00000000 n = 15 sin(180) = 0.00000000 n = 16 sin(180) = 0.00000000 n = 17 sin(180) = 0.00000000 n = 18 sin(180) = 0.00000000 n = 19 sin(180) = 0.00000000 n = 20 sin(180) = 0.00000000 </pre>

Observe that eight place accuracy was achieved by the tenth term of the series. However, it is still an imprecise approximation.

Here are four discreet rational returns for the cosine function.

<pre> Enter number of terms: 20 Enter an angle in deg: 60 n = 1 cos(60) = 0.52359878 n = 2 cos(60) = 0.49967418 n = 3 cos(60) = 0.50000213 n = 4 cos(60) = 0.49999999 n = 5 cos(60) = 0.50000000 n = 6 cos(60) = 0.50000000 n = 7 cos(60) = 0.50000000 n = 8 cos(60) = 0.50000000 n = 9 cos(60) = 0.50000000 n = 10 cos(60) = 0.50000000 n = 11 cos(60) = 0.50000000 n = 12 cos(60) = 0.50000000 n = 13 cos(60) = 0.50000000 n = 14 cos(60) = 0.50000000 n = 15 cos(60) = 0.50000000 n = 16 cos(60) = 0.50000000 n = 17 cos(60) = 0.50000000 n = 18 cos(60) = 0.50000000 n = 19 cos(60) = 0.50000000 n = 20 cos(60) = 0.50000000 </pre>	<pre> Enter number of terms: 20 Enter an angle in deg: 90 n = 1 cos(90) = 0.00000000 n = 2 cos(90) = 0.00000000 n = 3 cos(90) = 0.00000000 n = 4 cos(90) = 0.00000000 n = 5 cos(90) = 0.00000000 n = 6 cos(90) = 0.00000000 n = 7 cos(90) = 0.00000000 n = 8 cos(90) = 0.00000000 n = 9 cos(90) = 0.00000000 n = 10 cos(90) = 0.00000000 n = 11 cos(90) = 0.00000000 n = 12 cos(90) = 0.00000000 n = 13 cos(90) = 0.00000000 n = 14 cos(90) = 0.00000000 n = 15 cos(90) = 0.00000000 n = 16 cos(90) = 0.00000000 n = 17 cos(90) = 0.00000000 n = 18 cos(90) = 0.00000000 n = 19 cos(90) = 0.00000000 n = 20 cos(90) = 0.00000000 </pre>
<pre> Enter number of terms: 20 Enter an angle in deg: 120 n = 1 cos(120) = -0.52359878 n = 2 cos(120) = -0.49967418 n = 3 cos(120) = -0.50000213 n = 4 cos(120) = -0.49999999 n = 5 cos(120) = -0.50000000 n = 6 cos(120) = -0.50000000 n = 7 cos(120) = -0.50000000 n = 8 cos(120) = -0.50000000 n = 9 cos(120) = -0.50000000 n = 10 cos(120) = -0.50000000 n = 11 cos(120) = -0.50000000 n = 12 cos(120) = -0.50000000 n = 13 cos(120) = -0.50000000 n = 14 cos(120) = -0.50000000 n = 15 cos(120) = -0.50000000 n = 16 cos(120) = -0.50000000 n = 17 cos(120) = -0.50000000 n = 18 cos(120) = -0.50000000 n = 19 cos(120) = -0.50000000 n = 20 cos(120) = -0.50000000 </pre>	<pre> Enter number of terms: 20 Enter an angle in deg: 180 n = 1 cos(180) = -1.57079633 n = 2 cos(180) = -0.92483223 n = 3 cos(180) = -1.00452486 n = 4 cos(180) = -0.99984310 n = 5 cos(180) = -1.00000354 n = 6 cos(180) = -0.99999994 n = 7 cos(180) = -1.00000000 n = 8 cos(180) = -1.00000000 n = 9 cos(180) = -1.00000000 n = 10 cos(180) = -1.00000000 n = 11 cos(180) = -1.00000000 n = 12 cos(180) = -1.00000000 n = 13 cos(180) = -1.00000000 n = 14 cos(180) = -1.00000000 n = 15 cos(180) = -1.00000000 n = 16 cos(180) = -1.00000000 n = 17 cos(180) = -1.00000000 n = 18 cos(180) = -1.00000000 n = 19 cos(180) = -1.00000000 n = 20 cos(180) = -1.00000000 </pre>

Observe that eight place accuracy was achieved by the fifth term of the series. However, it is still an imprecise approximation.

Now let us consider the arc-sine and the arc-cosine. This is the inversion of the preceding. Here is a Taylor infinite series that approaches the angle represented by $[y/r]$. This is the arc-sine. The arc-cosine is found by first solving for the arc-sine and then phase shifting it 90° ($\pi/2$). The return is expressed in terms of radians.

$$\text{asin}(x) = x + \left(\frac{1}{2}\right) \cdot \frac{x^3}{3} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right) \cdot \frac{x^5}{5} + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right) \cdot \frac{x^7}{7} + \dots$$

$$\text{acos}(x) = \frac{\pi}{2} - \text{asin}(x)$$

Here is a LIBERTY BASIC v4.09 program that was written to evaluate this series.

```

REM LIBERTY BASIC v4.03
REM TaylorArcSinCos.bas
INPUT "Enter number of terms: ";t
INPUT "Enter number -1<=x<=1: ";x
LET p = 3.14159265
LET n = 1
LET a = x
LET b = a * (180/p)
LET c = (p/2) - a
LET d = c * (180/p)
PRINT "n = "; using("###",n); " sin(x) = "; using("###.#####",b);
PRINT " cos(x) = "; using("###.#####",d)
LET n = 2
LET e = 1
LET f = 2
LET g = 3
LET h = e/f
WHILE n <= t
LET a = a + h * (x^g/g)
LET b = a * (180/p)
LET c = (p/2) - a
LET d = c * (180/p)
PRINT "n = "; using("#####",n); " sin(x) = "; using("###.#####",b);
PRINT " cos(x) = "; using("###.#####",d)
LET n = n + 1
LET e = e + 2
LET f = f + 2
LET g = g + 2
LET h = h * (e/f)
WEND
END

```

Let us consider the case of $\sin(\theta) = 0.5$. This is a known value. The arc-sine is equal to 30° or $\pi/6$. Now let us examine what happens over 20 terms.

Enter number of terms: 20

Enter number $-1 \leq x \leq 1$: .5

n = 1	sin(x) =	28.64788979	cos(x) =	61.35211021
n = 2	sin(x) =	29.84155186	cos(x) =	60.15844814
n = 3	sin(x) =	29.97583885	cos(x) =	60.02416115
n = 4	sin(x) =	29.99582203	cos(x) =	60.00417797
n = 5	sin(x) =	29.99922195	cos(x) =	60.00077805
n = 6	sin(x) =	29.99984784	cos(x) =	60.00015216
n = 7	sin(x) =	29.99996921	cos(x) =	60.00003079
n = 8	sin(x) =	29.99999362	cos(x) =	60.00000638
n = 9	sin(x) =	29.99999867	cos(x) =	60.00000133
n = 10	sin(x) =	29.99999974	cos(x) =	60.00000026
n = 11	sin(x) =	29.99999997	cos(x) =	60.00000003
n = 12	sin(x) =	30.00000002	cos(x) =	59.99999998
n = 13	sin(x) =	30.00000003	cos(x) =	59.99999997
n = 14	sin(x) =	30.00000003	cos(x) =	59.99999997
n = 15	sin(x) =	30.00000003	cos(x) =	59.99999997
n = 16	sin(x) =	30.00000003	cos(x) =	59.99999997
n = 17	sin(x) =	30.00000003	cos(x) =	59.99999997
n = 18	sin(x) =	30.00000003	cos(x) =	59.99999997
n = 19	sin(x) =	30.00000003	cos(x) =	59.99999997
n = 20	sin(x) =	30.00000003	cos(x) =	59.99999997

As may be clearly seen, at 12 terms it is reasonably near what it should be, but then it pushes beyond very slightly.

Let us look at the case of 45° or $\pi/4$. $\sin(\pi/4) = 0.70710678$

Enter number of terms: 20

Enter number $-1 \leq x \leq 1$: 0.70710678

n = 1	sin(x) =	40.51423421	cos(x) =	49.48576579
n = 2	sin(x) =	43.89042038	cos(x) =	46.10957962
n = 3	sin(x) =	44.65006226	cos(x) =	45.34993774
n = 4	sin(x) =	44.87614616	cos(x) =	45.12385384
n = 5	sin(x) =	44.95307748	cos(x) =	45.04692252
n = 6	sin(x) =	44.98140220	cos(x) =	45.01859780
n = 7	sin(x) =	44.99238710	cos(x) =	45.00761290
n = 8	sin(x) =	44.99680722	cos(x) =	45.00319278
n = 9	sin(x) =	44.99863539	cos(x) =	45.00136461
n = 10	sin(x) =	44.99940782	cos(x) =	45.00059218
n = 11	sin(x) =	44.99973978	cos(x) =	45.00026022
n = 12	sin(x) =	44.99988444	cos(x) =	45.00011556
n = 13	sin(x) =	44.99994821	cos(x) =	45.00005179
n = 14	sin(x) =	44.99997660	cos(x) =	45.00002340
n = 15	sin(x) =	44.99998934	cos(x) =	45.00001066
n = 16	sin(x) =	44.99999511	cos(x) =	45.00000489

```

n = 17 sin(x) = 44.99999773 cos(x) = 45.00000227
n = 18 sin(x) = 44.99999893 cos(x) = 45.00000107
n = 19 sin(x) = 44.99999948 cos(x) = 45.00000052
n = 20 sin(x) = 44.99999973 cos(x) = 45.00000027

```

This too at 20 terms is approaching what it should be. Let us examine the case for 60° or $\pi/3$. $\sin(\pi/3) = 0.86602540$

```

Enter number of terms: 20
Enter number -1<=x<=1: 0.86602540
n = 1 sin(x) = 49.61960043 cos(x) = 40.38039957
n = 2 sin(x) = 55.82205043 cos(x) = 34.17794957
n = 3 sin(x) = 57.91537728 cos(x) = 32.08462272
n = 4 sin(x) = 58.84989819 cos(x) = 31.15010181
n = 5 sin(x) = 59.32689324 cos(x) = 30.67310676
n = 6 sin(x) = 59.59032459 cos(x) = 30.40967541
n = 7 sin(x) = 59.74357071 cos(x) = 30.25642929
n = 8 sin(x) = 59.83606569 cos(x) = 30.16393431
n = 9 sin(x) = 59.89344998 cos(x) = 30.10655002
n = 10 sin(x) = 59.92981854 cos(x) = 30.07018146
n = 11 sin(x) = 59.95326326 cos(x) = 30.04673674
n = 12 sin(x) = 59.96858805 cos(x) = 30.03141195
n = 13 sin(x) = 59.97872157 cos(x) = 30.02127843
n = 14 sin(x) = 59.98548808 cos(x) = 30.01451192
n = 15 sin(x) = 59.99004422 cos(x) = 30.00995578
n = 16 sin(x) = 59.99313431 cos(x) = 30.00686569
n = 17 sin(x) = 59.99524338 cos(x) = 30.00475662
n = 18 sin(x) = 59.99669094 cos(x) = 30.00330906
n = 19 sin(x) = 59.99768939 cos(x) = 30.00231061
n = 20 sin(x) = 59.99838113 cos(x) = 30.00161887

```

This looks reasonable as well. At 20 terms evaluated it is a close approximation.

However, when $x = 1$ the whole sequence stands on end. The sequence still returns the correct answer, but the number of required terms becomes astronomical! Observe that at 90° or $\pi/2$, $\sin(\pi/2) = 1$. Here is an abridged version of the run.

```

Enter number of terms: 99999
Enter number -1<=x<=1: 1
n = 1 sin(x) = 57.29577958 cos(x) = 32.70422042
n = 11 sin(x) = 80.21896051 cos(x) = 9.78103949
n = 111 sin(x) = 86.93063782 cos(x) = 3.06936218
n = 1111 sin(x) = 89.03014480 cos(x) = 0.96985520
n = 11111 sin(x) = 89.69332907 cos(x) = 0.30667093
n = 99999 sin(x) = 89.89777677 cos(x) = 0.10222323

```

This return is not acceptable by any means!

Using this particular Taylor series, the only way to obtain a reasonable degree of precision is to break the angle into $\pm 45^\circ$ sections and exchange the sine for the cosine and vice versa. Thus, for the arc-sine when $x = 0.86602540$, you take the square root of $(1 - 0.86602540^2)^{1/2} = 0.5$, use $x = 0.5$ instead, and treat it as the cosine. The desired sine is now $\pi/2 - \text{asin}(0.5)$ or $90^\circ - 30^\circ = 60^\circ$.

Now we come to the infinite Taylor series for the arc-tangent. This series is in two parts. One part is for a tangent that is less than or equal to one and the other part is for a tangent that is greater than one. This combination essentially does a $\pm 45^\circ$ degree breakdown as previously discussed. The return of course is expressed in terms of radians.

$$\text{atan}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad |x| \leq 1$$

$$\text{atan}(x) = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3 \cdot x^3} - \frac{1}{5 \cdot x^5} + \frac{1}{7 \cdot x^7} - \dots \quad |x| > 1 \quad x \neq 0$$

```

REM LIBERTY BASIC v4.03 program
REM TaylorTan.bas
INPUT "Enter number of terms: "; t
INPUT "Enter the value of the tangent: "; x
LET p = 3.14159265
IF x >= -1 AND x <= 1 THEN
LET n = 1
LET c = 0
WHILE n <= t
LET a = 2 * n - 1
LET b = -1^(n + 1)
LET c = c + b * (x^a/a)
LET d = c * (180/p)
PRINT "n = "; using("#####",n); " atan("; using("#.#####",x); ") = "; using("###.#####",d)
LET n = n + 1
WEND
END IF
IF x < -1 OR x > 1 THEN
LET n = 1
LET c = p/2
LET d = c * (180/p)
PRINT "n = "; using("#####",n); " atan("; using("###.#####",x); ") = "; using("###.#####",d)
LET n = 2
WHILE n <= t
LET a = 2 * n - 3
LET b = -1^(n + 1)
LET c = c + b * (1/(a * x^a))
LET d = c * (180/p)
PRINT "n = "; using("#####",n); " atan("; using("###.#####",x); ") = "; using("###.#####",d)
LET n = n + 1
WEND
END IF

```

The preceding Taylor infinite series is followed by a LIBERTY BASIC 4.03 program. Let us now take a look at some predefined sample runs.

```

Enter number of terms:          20
Enter the value of the tangent: 0
n =  1  atan(0.00000000) =  0.00000000
n =  2  atan(0.00000000) =  0.00000000
n =  3  atan(0.00000000) =  0.00000000
n =  4  atan(0.00000000) =  0.00000000
n =  5  atan(0.00000000) =  0.00000000
n =  6  atan(0.00000000) =  0.00000000
n =  7  atan(0.00000000) =  0.00000000
n =  8  atan(0.00000000) =  0.00000000
n =  9  atan(0.00000000) =  0.00000000
n = 10  atan(0.00000000) =  0.00000000
n = 11  atan(0.00000000) =  0.00000000
n = 12  atan(0.00000000) =  0.00000000
n = 13  atan(0.00000000) =  0.00000000
n = 14  atan(0.00000000) =  0.00000000
n = 15  atan(0.00000000) =  0.00000000
n = 16  atan(0.00000000) =  0.00000000
n = 17  atan(0.00000000) =  0.00000000
n = 18  atan(0.00000000) =  0.00000000
n = 19  atan(0.00000000) =  0.00000000
n = 20  atan(0.00000000) =  0.00000000

```

This looks good. The tangent of 0 is always by definition exactly 0. Let us do the same with one.

```

Enter number of terms:          20
Enter the value of the tangent: 1
n =  1  atan(1.00000000) =  57.29577958
n =  2  atan(1.00000000) =  38.19718639
n =  3  atan(1.00000000) =  49.65634230
n =  4  atan(1.00000000) =  41.47123093
n =  5  atan(1.00000000) =  47.83742866
n =  6  atan(1.00000000) =  42.62872143
n =  7  atan(1.00000000) =  47.03608909
n =  8  atan(1.00000000) =  43.21637045
n =  9  atan(1.00000000) =  46.58671043
n = 10  atan(1.00000000) =  43.57114308
n = 11  atan(1.00000000) =  46.29951354
n = 12  atan(1.00000000) =  43.80839268
n = 13  atan(1.00000000) =  46.10022387
n = 14  atan(1.00000000) =  43.97815796
n = 15  atan(1.00000000) =  45.95387449
n = 16  atan(1.00000000) =  44.10562354
n = 17  atan(1.00000000) =  45.84185929
n = 18  atan(1.00000000) =  44.20483701
n = 19  atan(1.00000000) =  45.75337159
n = 20  atan(1.00000000) =  44.28424904

```

Notice how the results go alternately high and low while the differences become smaller and smaller. At a certain number of terms the last two results could be averaged to produce an apparent precision result within the desired range. By definition the arc-tangent of one is always $\pi/4$ rad or 45° . Now let us consider two irrational cases that are not defined.

```
Enter number of terms:          20
Enter the value of the tangent: 0.1
n =  1  atan(0.10000000) =  5.72957796
n =  2  atan(0.10000000) =  5.71047936
n =  3  atan(0.10000000) =  5.71059396
n =  4  atan(0.10000000) =  5.71059314
n =  5  atan(0.10000000) =  5.71059314
n =  6  atan(0.10000000) =  5.71059314
n =  7  atan(0.10000000) =  5.71059314
n =  8  atan(0.10000000) =  5.71059314
n =  9  atan(0.10000000) =  5.71059314
n = 10  atan(0.10000000) =  5.71059314
n = 11  atan(0.10000000) =  5.71059314
n = 12  atan(0.10000000) =  5.71059314
n = 13  atan(0.10000000) =  5.71059314
n = 14  atan(0.10000000) =  5.71059314
n = 15  atan(0.10000000) =  5.71059314
n = 16  atan(0.10000000) =  5.71059314
n = 17  atan(0.10000000) =  5.71059314
n = 18  atan(0.10000000) =  5.71059314
n = 19  atan(0.10000000) =  5.71059314
n = 20  atan(0.10000000) =  5.71059314
```

As we can see, this run is quickly reduced to an eight place precision. Now let us look at the inversion of this entry where the tangent is flipped with cotangent. The entry was 0.1 so the inversion must be $1/0.1 = 10$.

```
Enter number of terms:          20
Enter the value of the tangent: 10
n =  1  atan( 10.00000000) = 90.00000000
n =  2  atan( 10.00000000) = 84.27042204
n =  3  atan( 10.00000000) = 84.28952064
n =  4  atan( 10.00000000) = 84.28940604
n =  5  atan( 10.00000000) = 84.28940686
n =  6  atan( 10.00000000) = 84.28940686
n =  7  atan( 10.00000000) = 84.28940686
n =  8  atan( 10.00000000) = 84.28940686
n =  9  atan( 10.00000000) = 84.28940686
n = 10  atan( 10.00000000) = 84.28940686
n = 11  atan( 10.00000000) = 84.28940686
n = 12  atan( 10.00000000) = 84.28940686
```

```

n = 13 atan( 10.00000000) = 84.28940686
n = 14 atan( 10.00000000) = 84.28940686
n = 15 atan( 10.00000000) = 84.28940686
n = 16 atan( 10.00000000) = 84.28940686
n = 17 atan( 10.00000000) = 84.28940686
n = 18 atan( 10.00000000) = 84.28940686
n = 19 atan( 10.00000000) = 84.28940686
n = 20 atan( 10.00000000) = 84.28940686

```

As can be seen, this too quickly settles down to an apparent acceptable eight place precision.

There are many other transcendental equations and many more computer programs to work them out to a desired degree of precision. However, they are still imprecise!

In our current age of electronic computers, we lose site of the complexity of the machines and the exotic processes in the manufacture of the components. We forget that a mere 50 years ago our integrated circuits (IC) of MOSFETS and DIODES were being done by much larger printed circuits (PC) were being used in basement buildings kept at a temperature of 50°F. Before that, at the beginning of the computer age, the computers used electro-mechanical modules. This latter was a mere 80 years ago.

A computer at its core is nothing more than a binary abacus. The largest number that may be processed is determined by the bit rating of the processor. For example, a 16 bit processor will have a capacity of $2^{16} - 1 = 32,768$ or four base 10 digits. This reminiscent of the first calculators. On the other hand, a 32 bit system will handle $2^{32} = 1,073,741,824$ or nine base 10 digits. Today, we have computers using 64 bit processors that can handle $2^{64} = 1,152,921,504,606,846,976$ or eighteen base 10 digits. However, even this latter cannot work out pi (π) to the hundredth place!

A modern computer has three processors. There is a BIOS that is used to access the peripheral hardware. There is the main processor. Finally, there is a math coprocessor which is a glorified calculator

chip. The math coprocessor has the most basic transcendental programs hardwired and often has pi (π) to the 6th or 8th place hardwired as well. The other transcendental programs are introduced as software as needed in sub-programs or header files.

Conclusion

This document is not intended as a primer in computers or mathematics. Its intended purpose is to reveal the instability of the technology and the errors in assuming that the apparent resulting precision is always correct. This all goes back to the rule of significant digits. The precision of the final solution of any series of equations is no greater than the element with the least number of significant digits!

The modern computer technology will fail due to its innate instabilities. It is too exotic and too subject to social, political, and economic whims. It is too dependent upon consumerism!

Before we had computers, we had the abacus and the slide rule. We had mechanical counters. Foremost of all; we had mathematics in the forms of addition, subtraction, multiplication, division, trigonometry, algebra, and calculus.

In writing the programs in this document, I have written them in the BASIC language which is an English language interpretation of the Boolean language of logic that came about long before computers. No computer is needed to work these programs. However, without the computer, any of these series will soon become very tedious.

Wed 29 Nov 2023

Patrick Richard Ahmatov